ON THE THEORY OF STRONG INTERACTION OF THE BOUNDARY LAYER WITH AN INVISCID HYPERSONIC FLOW

(K TEORII SIL'NOGO VZAIMODEISTVIIA POGRANICHNOGO SLOIA S NEVIAZKIM GIPERZVUKOVYM POTOKOM)

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N.S.MATVEEVA and V.V.SYCHEV
(Moscow)

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We shall consider higher approximations in the theory of strong interaction of a boundary layer with an external inviscid flow. We refine known results related to the problems of unsteady gas flow near an infinite plate and steady flow past a semi-infinite plate (Sections 1 to 6). As a result the asymptotic representations for the transverse displacement of the plate, or its form are found, corresponding to a pressure distribution law of a first approximation.

The influence of viscosity and thermal conductivity of the gas on the flow field near the body moving with hypersonic speed, as is well known, may be approximately considered on the basis of the theory of interaction of the boundary layer with the external inviscid flow region [1] . If, moreover, the body is sufficiently slender, and the Mach number M_∞ and the Reynolds number R_∞ of the problem are such that the ratio $M_\infty^3 / \sqrt{R_\infty} \gg 1$, then the phenomenon of strong interaction occurs, in which the pressure field in the perturbed flow region is mainly determined by the displacement effect of the boundary layer and depends to a considerably less extent on the form of the body surface. Examples of plane flows of this type have been considered in [2 and 3].

The construction of the solutions in these papers were based on the matching of the exact (self-similar) solutions of the equations of the boundary layer and of the equations of small perturbation theory in hypersonic flow. The matching process of these solutions was carried out only to a first approximation. As a consequence of this, there appeared some special character in the behavior of the solution in the intermediate region (at the outer edge of the boundary layer), where the enthalpy of the gas tends to zero and the density increases without bound. In references [2 and 3], estimates of accuracy were carried out for the first approximation theory.

The present paper is entirely devoted to the construction of higher approximations for these problems; or more rigorously, for problems of the asymptotic behavior of the flow field of a viscous heat-conducting gas behind shock waves, propagating according to the same law ($y \sim t^{3/4}$ and $y^{3/4} \sim x^{3/4}$) in the limiting case of $M_\infty \to \infty$.

1. Let us consider the one-dimensional unsteady motion of a viscous heat-conducting gas under the action of an infinite plate, suddenly set to motion with a velocity having a constant longitudinal component U_∞ . We assume a linear relationship for the coefficient of viscosity and the specific enthalpy

$$\mu = CU_\infty^2 h \tag{1.1}$$

The Navier-Stokes equations for this case may be written as

$$\rho\left(\frac{\partial u}{\partial t} + v\,\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial y}\left(h\,\frac{\partial u}{\partial y}\right), \qquad \rho\left(\frac{\partial v}{\partial t} + v\,\frac{\partial v}{\partial y}\right) + \frac{\partial p}{\partial y} = \frac{4}{3}\,\frac{\partial}{\partial y}\left(h\,\frac{\partial v}{\partial y}\right)$$
$$\rho\left(\frac{\partial h}{\partial t} + v\,\frac{\partial h}{\partial y}\right) = \frac{\partial p}{\partial t} + v\,\frac{\partial p}{\partial y} + \frac{1}{\sigma}\,\frac{\partial}{\partial y}\left(h\,\frac{\partial h}{\partial y}\right) + h\left(\frac{\partial u}{\partial y}\right)^2 + \frac{4}{3}\,h\left(\frac{\partial v}{\partial y}\right)^2 \tag{1.2}$$
$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho v}{\partial y} = 0, \qquad p = \frac{\gamma - 1}{\gamma}\,\rho h$$

Here the velocity components u and v are taken relative to the longitudinal plate velocity U_∞ ; the pressure p — relative to the quantity $\rho_\infty U_\infty^2$; the density ρ — relative to the unperturbed flow density ρ_∞ ; the specific enthalpy h — relative to the quantity U_∞^2; the dimensionless independent variables t and y — relative to the quantities C/ρ_∞ and CU_∞/ρ_∞ , respectively; and finally, σ and γ are the Prandtl number and ratio of specific heats of the gas, respectively.

Introducing on the basis of the continuity equation the function ψ , defined by the relations

$$\partial\psi / \partial t = -\rho v, \qquad \partial\psi / \partial y = \rho \tag{1.3}$$

we transform system (1.2) to independent variables t and ψ . As a result, we have

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial \psi}\left(\rho h\,\frac{\partial u}{\partial \psi}\right), \qquad \frac{\partial v}{\partial t} + \frac{\partial p}{\partial \psi} = \frac{4}{3}\,\frac{\partial}{\partial \psi}\left(\rho h\,\frac{\partial v}{\partial \psi}\right)$$
$$\rho\,\frac{\partial h}{\partial t} = \frac{\partial p}{\partial t} + \frac{1}{\sigma}\,\rho\,\frac{\partial}{\partial \psi}\left(\rho h\,\frac{\partial h}{\partial \psi}\right) + \rho^2 h\left(\frac{\partial u}{\partial \psi}\right)^2 + \frac{4}{3}\;{}^2h\left(\frac{\partial v}{\partial \psi}\right)^2 \tag{1.4}$$
$$\rho\,\frac{\partial y}{\partial \psi} = 1, \qquad \frac{\partial y}{\partial t} = v, \qquad p = \frac{\gamma - 1}{\gamma}\,\rho h$$

The purpose of this paper, as already indicated, is the construction of asymptotic solution to these equations, corresponding to the one-dimensional motion of a gas behind a shock wave propagating according to the law

$$y = ct^{3/4} \tag{1.5}$$

The solution is to satisfy the no-slipping condition

$$u = 1 \tag{1.6}$$

and the no-heat-flow condition

$$\partial h / \partial \psi = 0 \tag{1.7}$$

on the plate surface $\psi = 0$. Thus, the plate is assumed to be insulated.

2. For the external part of the flow field, adjacent to the shock wave, the solution has the well known form

$$y = t^{3/4}Y_0(\nu), \qquad u = 0, \qquad v = t^{-1/4}V_0(\nu)$$
$$p = t^{-1/2}P_0(\nu), \qquad \rho = R_0(\nu), \qquad h = t^{-1/2}H_0(\nu) \tag{2.1}$$

where the independent variable is

$$\nu = \psi t^{-3/4} \tag{2.2}$$

Substituting Expressions (2.1) in (1.4) and keeping the dominant terms in these equations, we find the system of equations for the well-known self-similar motion of an inviscid gas

$$\begin{gathered}{}^3/_4\nu V_0' + {}^1/_4 V_0 = P_0', \qquad R_0({}^3/_2\nu H_0' + H_0) = {}^3/_2\nu P_0' + P_0 \\ R_0 Y_0' = 1, \quad {}^3/_4\nu Y_0' - {}^3/_4 Y_0 + V_0 = 0, \quad P_0 = [(\gamma-1)/\gamma]\, R_0 H_0\end{gathered} \qquad (2.3)$$

We note that considering the gas in the outer flow region as inviscid and non-heat-conducting is correct with a relative error of order t^{-1}, since the ratio of the neglected terms in (1.4) to the dominant terms is of this order.

The solution to system (2.3) must satisfy the system of boundary conditions on the surface of the shock wave, which is propagating according to Equation (1.5). In the limiting case of flow with $M_\infty \to \infty$, these boundary conditions assume the form $\nu_0(c) = c$

$$V_0(c) = \frac{3c}{2(\gamma+1)}, \quad P_0(c) = \frac{9c^2}{8(\gamma+1)}, \quad R_0(c) = \frac{\gamma+1}{\gamma-1}, \quad H_0(c) = \frac{9\gamma c^2}{8(\gamma+1)} \qquad (2.4)$$

Here the constant c is to be determined.

For further use, it is important to have a representation of the desired functions of the external flow for $\nu \to 0$. To obtain these expressions, we note that the second equation in (2.2) can be integrated with the help of the last equation to yield

$$P_0 R_0^{-\gamma} = A_0 \nu^{-2/3} \qquad (2.5)$$

The constant A_0 is determined from the boundary condition (2.4)

$$A_0 = \frac{9c^{4/3}}{8(\gamma+1)}\left(\frac{\gamma-1}{\gamma+1}\right)^\gamma \qquad (2.6)$$

We use (2.5) and the remaining equations of the system (2.3); now we obtain without difficulty the following expressions, valid for $\nu \to 0$:

$$\begin{gathered}Y_0 = Y_{00} + Y_{01}\nu^{1-2/3\gamma} + O(\nu^{2-2/3\gamma}), \qquad R_0 = R_{00}\nu^{2/3\gamma} + O(\nu^{1+2/3\gamma}) \\ V_0 = V_{00} + V_{01}\nu^{1-2/3\gamma} + O(\nu^{2-2/3\gamma}), \qquad H_0 = H_{00}\nu^{-2/3\gamma} + O(\nu^{1-2/3\gamma}) \\ P_0 = P_{00} + O(\nu)\end{gathered} \qquad (2.7)$$

The coefficients in these formulas are connected by the relations

$$\begin{gathered}Y_{01} = \frac{3\gamma}{3\gamma-2} A^{1/\gamma} P_{00}^{-1/\gamma}, \quad V_{00} = \frac{3}{4} Y_{00}, \quad V_{01} = \frac{3}{2(3\gamma-2)} A^{1/\gamma} P_{01}^{-1/\gamma} \\ R_{00} = A_0^{-1/\gamma} P_{00}^{1/\gamma}, \qquad H_{00} = \frac{\gamma}{\gamma-1} A_0^{1/\gamma} P_{00}^{1-1/\gamma}\end{gathered} \qquad (2.8)$$

3. To study the interior region of the flow field, we introduce (as usually done) the independent variable

$$N = \psi t^{-1/4} \qquad (3.1)$$

To determine the asymptotic expansions, valid in this region, we express the functions of the external flow in terms of the independent variable of the inner expansion

$$\nu = N t^{-1/2} \qquad (3.2)$$

and pass to the limit $t \to \infty$ for fixed value of N. Using expression (2.7), we get

$$y = t^{3/4}[Y_{00} + Y_{01}N^{1-2/3\gamma}t^{-1/2+1/3\gamma} + O(t^{-1+1/3\gamma})], \quad u = O(t^{-1})$$
$$v = t^{-1/4}[V_{00} + V_{01}N^{1-2/3\gamma}t^{-1/2+1/3\gamma} + O(t^{-1+1/3\gamma})], \quad p = t^{-1/2}[P_{00} + O(t^{-1/2})]$$
$$\rho = R_{00}N^{2/3\gamma}t^{-1/3\gamma} + O(t^{-1/2-1/3\gamma}), \qquad h = H_{00}N^{-2/3\gamma}t^{-1/2+1/3\gamma} + O(t^{-1+1/3\gamma}) \tag{3.3}$$

These expressions suggest the form in which to seek the asymptotic solution for the inner flow region, thus

$$y = t^{3/4}[y_0(N) + t^{-1/2+1/3\gamma}y_1(N) + \cdots], \quad u = u_0(N) + t^{-1/2+1/3\gamma}u_1(N) + \cdots]$$
$$v = t^{-1/4}[v_0(N) + t^{-1/2+1/3\gamma}v_1(N) + \cdots]$$
$$p = t^{-1/2}[p_0(N) + t^{-1/2+1/3\gamma}p_1(N) + \cdots] \tag{3.4}$$
$$\rho = t^{-1/2}[\rho_0(N) + t^{-1/2+1/3\gamma}\rho_1(N) + \cdots], \qquad h = h_0(N) + t^{-1/2+1/3\gamma}h_1(N) + \cdots$$

In fact, the matching of the inner and outer expansions will now be guaranteed, if in accordance with the simple form of the matching principle [4], the following boundary conditions are satisfied for the function of inner expansion at $N \to \infty$:

in the first approximation

$$y_0(N) \to Y_{00}, \qquad u_0(N) \to 0, \qquad p_0(N) \to P_{00}, \qquad h_0(N) \to 0 \tag{3.5}$$

in the second approximation

$$y_1(N) \to Y_{01}N^{1-2/3\gamma}, \quad u_1(N) \to 0, \quad p_1(N) \to 0, \quad h_1(N) \to H_{00}N^{-2/3\gamma} \tag{3.6}$$

4. Substituting into the initial equations (1.4) the expansions (3.4) and keeping the main terms, we obtain the system of equations for the first approximation, which may be written in the form

$$p_0 = \frac{\gamma-1}{\gamma}\rho_0 h_0 = \text{const}, \qquad u_0'' + \frac{\gamma-1}{4\gamma p_0}Nu_0' = 0$$
$$\frac{\gamma}{\gamma-1}\frac{p_0}{\sigma}h_0'' + \frac{1}{4}Nh_0' - \frac{\gamma-1}{2\gamma}h_0 = -\frac{\gamma}{\gamma-1}p_0 u_0'^2 \tag{4.1}$$
$$y_0' = \frac{\gamma-1}{\gamma}\frac{h_0}{p_0}, \qquad v_0 = \frac{3}{4}y_0 - \frac{\gamma-1}{4\gamma}N\frac{h_0}{p_0}$$

Boundary conditions for these equations are (3.5), and also conditions on the surface of the plate in the form

$$u_0(0) = 1, \qquad y_0(0) = h_0'(0) = 0 \tag{4.2}$$

i.e. besides the satisfaction of boundary conditions (1.6) and (1.7) we require that the plate in the first approximation be moved in its own plane. The formulation of the problem in the first approximation completely agrees with the problem considered in [2]. Its solution turns out to be quite simple. First, we note that the second equation in (4.1) integrates by a quadrature. Its particular solution, satisfying the boundary conditions, is

$$u_0 = 1 - \left(\frac{\gamma-1}{2\pi\gamma p_0}\right)^{1/2}\int_0^N \exp\left[-\frac{\gamma-1}{8\gamma p_0}N^2\right]dN \tag{4.3}$$

We can then integrate the third equation. However, for determining the pressure distribution on the surface of the plate there is no need to do this. For this problem, it suffices to find the expression for $y_0(N)$ for $N \to \infty$. On the basis of the fourth equation in (4.1) we have

$$\lim_{N\to\infty} y_0(N) = \frac{\gamma-1}{\gamma p_0}\int_0^\infty h_0 dN \tag{4.4}$$

The integral in this expression is easily calculated with the help of the third equation in (4.1), if we take into account the boundary condition for $h_0(N)$ and the exponential decay to zero of this function as $N \to \infty$ (cf.[2]). As a result, we get

$$\lim_{N\to\infty} y_0(N) = \frac{2\gamma}{3\gamma-2}\left(\frac{\gamma-1}{\pi\gamma h_0}\right)^{1/2} \tag{4.5}$$

Using boundary conditions (3.5), we rewrite this as

$$Y_{00} = \frac{2\gamma}{3\gamma-2}\left(\frac{\gamma-1}{\pi\gamma P_{00}}\right)^{1/2} \tag{4.6}$$

The obtained relation is just the boundary condition, which was missing for the equations of the outer inviscid flow. This condition uniquely determines the constant c in the equation of the shock wave (1.5) and in the boundary conditions (2.4), and therefore, it completes the problem in the first approximation.

5. Let us now turn to the problem of the second approximation. After substituting the expansions (3.4) in the system (1.4) and equating corresponding terms of the expansion, we obtain a system of linear differential equations for the functions in the second approximation.

The second and last equations in (1.4), together with the boundary conditions (3.6), give

$$p_1 = \frac{\gamma-1}{\gamma}(\rho_0 h_1 + h_0\rho_1) = 0 \tag{5.1}$$

After this, on the basis of the first equation in (1.4) and boundary condition (3.6) for u_1 we find that $u_1 = 0$.

Then the equation for determining the function h_1, after some simple transformations using (5.1) and the results of the first approximation, assumes the form

$$\frac{\gamma}{\gamma-1}\frac{p_0}{\sigma}h_1'' + \frac{1}{4}Nh_1' + \frac{1}{6\gamma}h_1 = 0 \tag{5.2}$$

Its solution must satisfy the last boundary condition in (3.6), and also condition (1.7) on the insulated surface

$$h_1(N) \to H_{00}N^{-2/3\gamma}, \quad h_1'(0) = 0 \quad \text{for } N\to\infty \tag{5.3}$$

Finally, the equation for function y_1 has the form

$$y_1' = \frac{\gamma-1}{\gamma}\frac{h_1}{p_0} \tag{5.4}$$

where the function $y_1(N)$ must satisfy the first boundary condition (*) in

*) See footnote on the next page.

(3.6).

Integrating successively (5.2) and (5.4), we find the value of the function $y_1(0)$, which determines the transverse displacement of the plate

$$y \approx t^{1/4+1/3\gamma} y_1(0) \tag{5.5}$$

We note, that in obtaining the equations of the second approximation, in the initial equations were neglected terms whose ratios to the terms kept were of the order of $t^{-1/2+1/3\gamma}$, while in the expansions (3.4), on the basis of (3.3), the highest order in the neglected terms was $t^{-1/2}$. Thus, the solution of the problem in the second approximation is correct with a relative error of the order $t^{-1+2/3\gamma}$ or $t^{-1/2}$, while the relative error of the first approximation is of order $t^{-1/2+1/3\gamma}$.

6. Numerical calculations were carried out for values of $\gamma = 1.4$ and $\sigma = 1.0$.

The system of equations (2.3) for the outer flow field was integrated with the aim of determining the constants in expressions (2.7) by the Runge-Kutta method. Equation (5.2) with boundary conditions (5.3), governing the enthalpy distribution in the boundary layer, was solved by an approximate iteration method. The calculation of the flow field in the inner region (integration of (4.1)) has not been carried out.

The constants c, which define the propagation of the shock wave, and p_0 the pressure variation on the surface of the plate, were found as follows: $c = 1.1082$, $p_0 = 0.3432$.

The value of the constant $y_1(0)$, determining the required transverse displacement of the plate was found to be $y_1(0) = 0.2362$.

7. The equations of plane steady flow of a viscous heat-conducting gas may be written in the following nondimensional form:

$$\begin{gathered}
\rho\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) + \frac{\partial p}{\partial x} = \frac{\partial}{\partial x}\left[h\left(\frac{4}{3}\frac{\partial u}{\partial x} - \frac{2}{3}\frac{\partial v}{\partial y}\right)\right] + \frac{\partial}{\partial y}\left[h\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right] \\
\rho\left(u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right) + \frac{\partial p}{\partial y} = \frac{\partial}{\partial y}\left[h\left(\frac{4}{3}\frac{\partial v}{\partial y} - \frac{2}{3}\frac{\partial u}{\partial x}\right)\right] + \frac{\partial}{\partial x}\left[h\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right] \\
\rho\left(u\frac{\partial h}{\partial x} + v\frac{\partial h}{\partial y}\right) = u\frac{\partial p}{\partial x} + v\frac{\partial p}{\partial y} + \frac{1}{\sigma}\frac{\partial}{\partial x}\left(h\frac{\partial h}{\partial x}\right) + \frac{1}{\sigma}\frac{\partial}{\partial y}\left(h\frac{\partial h}{\partial y}\right) + \\
+ 2h\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2\right] + h\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)^2 - \frac{2}{3}h\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)^2 \\
\frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} = 0, \qquad p = \frac{\gamma - 1}{\gamma}\rho h
\end{gathered} \tag{7.1}$$

Here the components of the velocity are taken relative to the velocity of the unperturbed stream U_∞, the pressure – relative to the quantity $\rho_\infty U_\infty^2$, (where $\frac{1}{2}\rho_\infty U_\infty^2$ is the dynamic pressure), the density – relative to the density of the unperturbed stream ρ_∞, and the specific enthalpy – relative to the quantity U_∞^2. The independent variables are taken relative to the characteristic length

$$L = CU_\infty / \rho_\infty \tag{7.2}$$

Here C is the proportionality constant in the relation between the viscosity coefficient and the enthalpy, which we again take to be linear (1.1).

*) It is easily verified that the asymptotic character of the behavior of the inner expansion functions as $N \to \infty$, prescribed by the boundary conditions (3.5) and (3.6), completely agrees with that which follows from a direct analysis of the differential equations for these functions.

Introducing the stream function ψ, defined by the relations

$$\partial\psi / \partial x = -\rho v, \qquad \partial\psi / \partial y = \rho u \tag{7.3}$$

we transform Equations (7.1) to the independent variables x and ψ

$$\begin{aligned}
&\rho u \frac{\partial u}{\partial x} + \frac{\partial p}{\partial x} - \rho v \frac{\partial p}{\partial \psi} = \left(\frac{\partial}{\partial x} - \rho v \frac{\partial}{\partial \psi}\right)\left[\frac{4}{3} h \left(\frac{\partial u}{\partial x} - \rho v \frac{\partial u}{\partial \psi}\right) - \frac{2}{3} \rho h u \frac{\partial v}{\partial \psi}\right] + \\
&\qquad + \rho u \frac{\partial}{\partial \psi}\left[\rho h u \frac{\partial u}{\partial \psi} + h\left(\frac{\partial v}{\partial x} - \rho v \frac{\partial v}{\partial \psi}\right)\right] \\
&\rho u \frac{\partial v}{\partial x} + \rho u \frac{\partial p}{\partial \psi} = \rho u \frac{\partial}{\partial \psi}\left[\frac{4}{3} \rho h u \frac{\partial v}{\partial \psi} - \frac{2}{3} h \left(\frac{\partial u}{\partial x} - \rho v \frac{\partial u}{\partial \psi}\right)\right] + \\
&\qquad + \left(\frac{\partial}{\partial x} - \rho v \frac{\partial}{\partial \psi}\right)\left[\rho h u \frac{\partial u}{\partial \psi} + h\left(\frac{\partial v}{\partial x} - \rho v \frac{\partial v}{\partial \psi}\right)\right] \\
&\rho u \frac{\partial h}{\partial x} - u \frac{\partial p}{\partial x} = \frac{1}{\sigma}\left(\frac{\partial}{\partial x} - \rho v \frac{\partial}{\partial \psi}\right)\left[h\left(\frac{\partial h}{\partial x} - \rho v \frac{\partial h}{\partial \psi}\right)\right] + \frac{1}{\sigma} \rho u \frac{\partial}{\partial \psi}\left(\rho h u \frac{\partial h}{\partial \psi}\right) + \\
&\qquad + 2h\left[\left(\frac{\partial u}{\partial x} - \rho v \frac{\partial u}{\partial \psi}\right)^2 + \left(\rho u \frac{\partial v}{\partial \psi}\right)^2\right] + h\left(\rho u \frac{\partial u}{\partial \psi} + \frac{\partial v}{\partial x} - \rho v \frac{\partial v}{\partial \psi}\right)^2 - \\
&\qquad - \frac{2}{3} h \left(\frac{\partial u}{\partial x} - \rho v \frac{\partial u}{\partial \psi} + \rho u \frac{\partial v}{\partial \psi}\right)^2 \\
&\rho u \frac{\partial u}{\partial \psi} = 1, \qquad u \frac{\partial y}{\partial x} = v, \qquad p = \frac{\gamma - 1}{\gamma} \rho h
\end{aligned} \tag{7.4}$$

The problem will consist in the construction of the asymptotic solution of these equations, corresponding to the steady flow of a gas behind a shock wave with the shape

$$y = cx^{3/4} \tag{7.5}$$

and satisfying boundary conditions on the thermally insulated semi-infinite surface $\psi = 0$, whose shape $y = f(x)$ is to be found. These conditions will have the form

$$u = v = 0, \qquad \frac{\partial h}{\partial \psi} = \frac{f'(x)\, \partial h / \partial x}{\rho\,[1 + f'^2(x)]} \tag{7.6}$$

8. We start with the asymptotic expansion, valid for the outer part of the flow, and again confine ourselves to the approximation in which this part of the flow may be treated as inviscid. We write the expansions in this region in the form

$$\begin{aligned}
y &= \xi^{3/4}\,[Y_0(\nu) + \xi^{-1/2} Y_1(\nu) + \cdots], \qquad u - 1 = \xi^{1/2}\,[U_0(\nu) + \xi^{-1/2} U_1(\nu) + \cdots] \\
v &= \xi^{-1/4}\,[V_0(\nu) + \xi^{-1/2} V_1(\nu) + \cdots], \qquad p = \xi^{-1/2}\,[P_0(\nu) + \xi^{-1/2} P_1(\nu) + \cdots] \\
\rho &= R_0(\nu) + \xi^{-1/2} R_1(\nu) + \cdots, \qquad h = \xi^{-1/2}\,[H_0(\nu) + \xi^{-1/2} H_1(\nu) + \cdots]
\end{aligned} \tag{8.1}$$

where the independent variables ξ and ν are defined by the relations

$$x = \xi, \qquad \psi = \xi^{3/4} n = \xi^{3/4}\,[\nu + \xi^{-1/2}\Psi_1(\nu) + \cdots] \tag{8.2}$$

Here the expansion is made in one of the independent variables in order to obtain (following the method of [5]) the solution to the external inviscid flow valid in the entire flow field, including the vicinity of the plate surface. This is necessary because in contrast to the problem considered in the first part of this paper, the first terms of expansions (8.1) would represent not the exact solution to the outer inviscid flow, but only its approximate solution, possessing singularities which are not characteristic for the exact solution when $\nu \to 0$

By (8.2) we obtain the following Formulas for the derivatives

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial \xi} - \frac{3}{4}\xi^{-1}\nu\frac{\partial}{\partial \nu} + \frac{3}{4}\xi^{-3/2}\left(\nu\Psi_1' - \frac{1}{3}\Psi_1\right)\frac{\partial}{\partial \nu} + \ldots$$

$$\frac{\partial}{\partial \psi} = \xi^{-3/4}\frac{\partial}{\partial \nu} - \xi^{-5/4}\Psi_1'\frac{\partial}{\partial \nu} - \ldots \tag{8.3}$$

The boundary conditions for the outer solution are the conditions on the shock wave (7.5), which in the limiting case $M_\infty \to \infty$ are

$$n = c, \quad y = c\xi^{3/4}, \quad u - 1 = -\frac{9c^2}{8(\gamma+1)}\xi^{-1/2}\left[1 - \frac{9}{16}c^2\xi^{-1/2} + O(\xi^{-1})\right]$$

$$v = \frac{3c}{2(\gamma+1)}\xi^{-1/4}\left[1 - \frac{9}{16}c^2\xi^{-1/2} + O(\xi^{-1})\right] \tag{8.4}$$

$$p = \frac{9c^2}{8(\gamma+1)}\xi^{-1/2}\left[1 - \frac{9}{16}c^2\xi^{-1/2} + O(\xi^{-1})\right]$$

$$\rho = \frac{\gamma+1}{\gamma-1}, \qquad h = \frac{9\gamma c^2}{8(\gamma+1)^2}\xi^{-1/2}\left[1 - \frac{9}{16}c^2\xi^{-1/2} + O(\xi^{-1})\right]$$

Substituting the expansions (8.1) and (8.3) into the initial system of equations (7.4) and the boundary conditions (8.4) and keeping the dominant terms, we obtain systems of differential equations and boundary conditions for the first approximation. They are completely equivalent to the problem of inviscid unsteady one-dimensional flow (2.1) and (2.4), considered in the first part of this paper. Thus, substituting the independent variable t by ξ, we may use the corresponding formulas of Section 2 without any change.

For the longitudinal component of the velocity in the first approximation we have

$$U_0 + \frac{V_0^2}{3} + \frac{\gamma}{\gamma-1}\frac{P_0}{R_0} = 0 \tag{8.5}$$

From this

$$U_0 = U_{00}\nu^{-2/3\gamma} + O(\nu^\circ) \quad \text{for } \nu \to 0 \quad \left(U_{00} = -\frac{\gamma}{\gamma-1}A_0^{1/\gamma}P_{00}^{1-1/\gamma}\right) \tag{8.6}$$

9. The equations of the second approximation, after some simple transformations using the relations of the first approximation, may be written as

$$U_1 + V_0V_1 + H_1 + {}^1/_2U_0^2 = 0$$

$${}^3/_4(\nu V_1)' - {}^3/_4(\nu\Psi_1' - {}^1/_3\Psi_1)V_0' = P_1' - \Psi_1'P_0' \tag{9.1}$$

$${}^3/_2\nu^2(P_1/P_0 - \gamma R_1/R_0)' + \nu(P_1/P_0 - \gamma R_1/R_0) = -(\nu\Psi_1' - {}^1/_3\Psi_1)$$

$$Y_1' + \frac{1}{R_0}\left(U_0 + \frac{R_1}{R_0}\right) = \Psi_1'Y_0'$$

$${}^3/_4\nu Y_1' - {}^1/_4Y_1 + V_1 - V_0U_0 = {}^3/_4(\nu\Psi_1' - {}^1/_3\Psi_1)Y_0'$$

$$P_1 = \frac{\gamma-1}{\gamma}(R_0H_1 + H_0R_1)$$

In order to eliminate in the second approximation the entropy singularities (as $\nu \to 0$) of the order higher than in the first approximation, we may follow the method of [5] and set in the fourth equation of (9.1)

$$\Psi_1' = U_0 + \frac{R_1}{R_0} \tag{9.2}$$

Then this equation becomes

$$Y_1' = 0 \tag{9.3}$$

Now, these two equations, together with the remaining equations of (9.1), form a closed system. The boundary conditions for these equations, by (8.4), can be written as

$$\Psi_1(c) = 0, \quad Y_1(c) = 0, \quad U_1(c) = \frac{81c^4}{128(\gamma+1)}, \quad V_1(c) = -\frac{27c^3}{32(\gamma+1)}$$
$$P_1(c) = -\frac{81\,c^4}{128(\gamma+1)}, \quad R_1(c) = 0, \quad H_1(c) = -\frac{81\gamma c^4}{128(\gamma+1)^2} \tag{9.4}$$

The first of these boundary conditions eliminates the shifting of streamlines at the vicinity of shock wave. Equation (9.3) together with condition (9.4) gives

$$Y_1(\nu) = 0 \tag{9.5}$$

The second equation in (9.1) may now be integrated. Its solution satisfying the boundary conditions (9.4) has the form

$$\frac{P_1}{P_0} - \gamma\frac{R_1}{R_0} = -\frac{9}{16}c^{8/3}\nu^{-2/3} - \frac{2}{3}\nu^{-1}\Psi_1(\nu) \tag{9.6}$$

The simultaneous consideration of Equations (9.1), (9.2), (9.5) and (9.6), together with the results of the first approximation (2.5) and (2.7), permits us to determine the behavior of the functions of the second approximation as $\nu \to 0$. Their approximate representation in this region will have the form

$$\Psi_1 = \Psi_{10}\nu^{1/3} + O(\nu^{1-2/3\gamma}), \qquad U_1 = U_{10}\nu^{-2/3-2/3\gamma} + O(\nu^{-4/3\gamma})$$
$$V_1 = V_{10}\nu^{-2/3\gamma} + O(\nu^{1-4/3\gamma}), \qquad P_1 = O(\nu^{1/3}) \tag{9.7}$$
$$R_1 = R_{10}\nu^{-2/3+2/3\gamma} + O(\nu^{0}), \qquad H_1 = H_{10}\nu^{-2/3-2/3\gamma} + O(\nu^{-4/3\gamma})$$

where the coefficients in these formulas are related with the coefficients of the functions of the first approximation (2.8) through the relations

$$\Psi_{10} = -\frac{27c^{8/3}}{16(2-\gamma)}, \qquad U_{10} = -\frac{9\gamma c^{8/3}}{16(\gamma-1)(2-\gamma)}A^{1/\gamma}P_{00}^{\;1-1/\gamma}$$
$$V_{10} = -\frac{3\gamma}{4(\gamma-1)}A_0^{\;1/\gamma}V_{00}P_{00}^{\;1-1/\gamma}, \qquad R_{10} = -\frac{9c^{8/3}}{16(2-\gamma)}A_0^{\;-1/\gamma}P_{00}^{\;1/\gamma}$$
$$H_{10} = \frac{6\gamma c^{8/3}}{16(\gamma-1)(2-\gamma)}A_0^{\;1/\gamma}P_{00}^{\;1-1/\gamma} \tag{9.8}$$

in which the constant A_0 is determined by Equation (2.6).

10. In the inner flow region the dimensionless independent variable of the order of unity is

$$N = \psi\xi^{-4/3} \tag{10.1}$$

To determine the form of the solution in this region, we express the functions of the outer flow in terms of the independent variable of the inner expansion

$$n = N\xi^{-1/3} \tag{10.2}$$

and we consider their behavior as $\xi \to \infty$ for fixed value of N. To this end, we first substitute in (10.2) the expansion for the independent variable n (8.2) and find the following relationship between the independent

By (8.2) we obtain the following formulas for the derivatives

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial \xi} - \frac{3}{4}\xi^{-1}\nu\frac{\partial}{\partial \nu} + \frac{3}{4}\xi^{-3/2}\left(\nu\Psi_1' - \frac{1}{3}\Psi_1\right)\frac{\partial}{\partial \nu} + \ldots$$

$$\frac{\partial}{\partial \psi} = \xi^{-3/4}\frac{\partial}{\partial \nu} - \xi^{-5/4}\Psi_1'\frac{\partial}{\partial \nu} - \ldots \qquad (8.3)$$

The boundary conditions for the outer solution are the conditions on the shock wave (7.5), which in the limiting case $M_\infty \to \infty$ are

$$n = c, \quad y = c\xi^{3/4}, \quad u - 1 = -\frac{9c^2}{8(\gamma+1)}\xi^{-1/2}\left[1 - \frac{9}{16}c^2\xi^{-1/2} + O(\xi^{-1})\right]$$

$$v = \frac{3c}{2(\gamma+1)}\xi^{-1/4}\left[1 - \frac{9}{16}c^2\xi^{-1/2} + O(\xi^{-1})\right] \qquad (8.4)$$

$$p = \frac{9c^2}{8(\gamma+1)}\xi^{-1/2}\left[1 - \frac{9}{16}c^2\xi^{-1/2} + O(\xi^{-1})\right]$$

$$\rho = \frac{\gamma+1}{\gamma-1}, \qquad h = \frac{9\gamma c^2}{8(\gamma+1)^2}\xi^{-1/2}\left[1 - \frac{9}{16}c^2\xi^{-1/2} + O(\xi^{-1})\right]$$

Substituting the expansions (8.1) and (8.3) into the initial system of equations (7.4) and the boundary conditions (8.4) and keeping the dominant terms, we obtain systems of differential equations and boundary conditions for the first approximation. They are completely equivalent to the problem of inviscid unsteady one-dimensional flow (2.1) and (2.4), considered in the first part of this paper. Thus, substituting the independent variable t by ξ, we may use the corresponding formulas of Section 2 without any change.

For the longitudinal component of the velocity in the first approximation we have

$$U_0 + \frac{V_0^2}{3} + \frac{\gamma}{\gamma-1}\frac{P_0}{R_0} = 0 \qquad (8.5)$$

From this

$$U_0 = U_{00}\nu^{-2/3\gamma} + O(\nu^\circ) \quad \text{for } \nu \to 0 \quad \left(U_{00} = -\frac{\gamma}{\gamma-1}A_0^{1/\gamma}P_{00}^{1-1/\gamma}\right) \qquad (8.6)$$

9. The equations of the second approximation, after some simple transformations using the relations of the first approximation, may be written as

$$U_1 + V_0V_1 + H_1 + {}^1/_2U_0^2 = 0$$

$${}^3/_4(\nu V_1)' - {}^3/_4(\nu\Psi_1' - {}^1/_3\Psi_1)V_0' = P_1' - \Psi_1'P_0' \qquad (9.1)$$

$${}^3/_2\nu^2(P_1/P_0 - \gamma R_1/R_0)' + \nu(P_1/P_0 - \gamma R_1/R_0) = -(\nu\Psi_1' - {}^1/_3\Psi_1)$$

$$Y_1' + \frac{1}{R_0}\left(U_0 + \frac{R_1}{R_0}\right) = \Psi_1'Y_0'$$

$${}^3/_4\nu Y_1' - {}^1/_4Y_1 + V_1 - V_0U_0 = {}^3/_4(\nu\Psi_1' - {}^1/_3\Psi_1)Y_0'$$

$$P_1 = \frac{\gamma-1}{\gamma}(R_0H_1 + H_0R_1)$$

In order to eliminate in the second approximation the entropy singularities (as $\nu \to 0$) of the order higher than in the first approximation, we may follow the method of [5] and set in the fourth equation of (9.1)

$$\Psi_1' = U_0 + \frac{R_1}{R_0} \qquad (9.2)$$

11. Substituting expansion (10.5) in the initial system of equations (7.4) and equatinf the main terms, we obtain a system of equations for the first approximation, which may be written as

$$p_0 = \frac{\gamma-1}{\gamma}\rho_0 h_0 = \text{const} \tag{11.1}$$

$$\frac{\gamma}{\gamma-1} p_0 u_0 (u_0 u_0')' + \frac{1}{4} N u_0 u_0' + \frac{\gamma-1}{2\gamma} h_0 = 0$$

$$\frac{\gamma}{\gamma-1} p_0 \left[u_0\left(\frac{h_0}{\sigma} + \frac{u_0^2}{2}\right)'\right]' + \frac{1}{4} N \left(h_0 + \frac{u_0^2}{2}\right)' = 0$$

$$\rho_0 u_0 y_0' = 1, \qquad v_0 = u_0 (3/4 y_0 - 1/4 N y_0')$$

Boundary conditions for these equations are (10.6) and also the conditions on the solid surface, which by virtue of (7.6) and (7.7), can be written as

$$y_0(0) = u_0(0) = h_0'(0) = 0 \tag{11.2}$$

i.e. we assume that in the first approximation the body is a semi-infinite flat plate. If the Prandtl number $\sigma = 1$, then the integral of the equation of heat influx satisfying the boundary conditions (10.6) and (11.2) will be

$$h_0 + 1/2 u_0^2 = 1/2 \tag{11.3}$$

Below we shall consider only this case. The momentum equation then reduces to the form

$$\frac{\gamma}{\gamma-1} p_0 u_0 (u_0 u_0')' + \frac{1}{4} N u_0 u_0' + \frac{1}{4}\frac{\gamma-1}{\gamma}(1 - u_0^2) = 0 \tag{11.4}$$

where in accordance with the third of the boundary conditions (10.6), $p_0 = P_{00}$.

Boundary conditions for (11.4) are the second conditions in (10.6) and (11.2) (*). After determining $u_0(N)$, the function $y_0(N)$ is found by integrating the fourth equation in (11.1), which with the aid of (11.3) and (11.2) gives

$$y_0 = \frac{\gamma-1}{2\gamma}\frac{1}{p_0}\int_0^N \frac{1-u_0^2}{u_0} dN \tag{11.5}$$

Finally, the first boundary condition in (10.6) leads to the relationship

$$Y_{00} = \frac{\gamma-1}{2\gamma}\frac{1}{P_{00}}\int_0^\infty \frac{1-u_0^2}{u_0} dN \tag{11.6}$$

---

*) We note that by introducing the variables

$$\eta = \frac{1}{2}\left(\frac{\gamma-1}{\gamma p_0}\right)^{1/2}\int_0^N \frac{dN}{u_0}, \qquad f_0 = \frac{1}{2}\left(\frac{\gamma-1}{\gamma p_0}\right)^{1/2} N$$

Equation (11.4) may be reduced to the well-know form

$$\frac{d^3 f_0}{d\eta^3} + f_0 \frac{d^2 f_0}{d\eta^2} + \frac{\gamma-1}{\gamma}\left[1 - \left(\frac{df_0}{d\eta}\right)^2\right] = 0$$

with the boundary conditions

$$f_0 = \frac{df_0}{d\eta} = 0 \quad \text{for } \eta = 0, \qquad \frac{df_0}{d\eta} \to 1 \quad \text{for } \eta \to \infty$$

in which the integrand is parametrically dependent on P_{00}. Thus (11.6) is the necessary boundary condition for the outer problem in the first approximation, relating the quantities Y_{00} and P_{00}. Thereby, it uniquely determines the constant c, i.e. the shape of the shock wave, and completely closes the system of relations in the first approximation. The problem of the flow past a semi-infinite plate in this formulation was solved in [3].

12. We now turn to the second and third approximations. First of all, by virtue of the second equation in (7.4) and the boundary condition (10.7), we have

$$p_1 = \frac{\gamma - 1}{\gamma}(\rho_0 h_1 + h_0 \rho_1) = 0 \qquad (12.1)$$

The third equation in (7.4), after some transformations and using the relations obtained for the first approximation, integrates into

$$h_1 + u_0 u_1 = 0 \qquad (12.2)$$

This solution satisfies boundary conditions (10.7), since $H_{00} + U_{00} = 0$ according to (2.8) and (8.7). It also satisfies to the neccessary order of the approximation the boundary conditions on the wall, which is easily verified by substituting the expansions (10.5) into (7.6).

Now the first of the momentum equations (7.4), after some manipulations, leads to the following equation for the function u_1:

$$\frac{\gamma}{\gamma - 1} p_0 (u_0 u_1)'' + \frac{1}{4} N u_1' - \left[\frac{\gamma - 1}{4\gamma} \frac{1 + u_0^2}{u_0^2} + \left(-\frac{1}{2} + \frac{1}{3\gamma}\right)\right] u_1 = 0 \qquad (12.3)$$

The boundary conditions for it are the second condition in (10.7) and the no-slip condition (7.6), i.e.

$$u_1(0) = 0, \quad u_1(N) \to U_{01} N^{-2/3\gamma} \quad \text{for } N \to \infty \qquad (12.4)$$

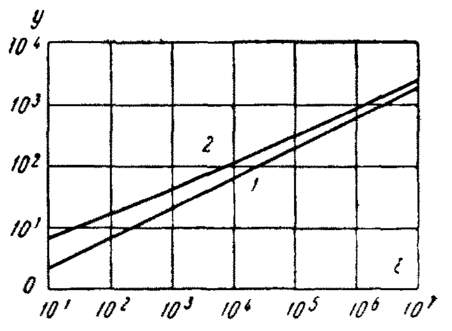

Fig. 1

Finally, the function $y_1(N)$ satisfies Equation

$$y_1' + \frac{\gamma - 1}{2\gamma p_0} \frac{1 + u_0^2}{u_0^2} u_1 = 0$$

The boundary condition for it is the first condition in (10.7) (*).

As a result of integrating (12.4) we find the value of the function $y_1(0)$ at the wall, defining the shape of the wall in the second approximation.

Similarly, we find the system of equations for the third approximation: integrals

$$p_2 = \frac{\gamma - 1}{\gamma}(\rho_0 h_2 + h_0 \rho_2) = 0, \qquad h_2 + u_0 u_2 = 0 \qquad (12.5)$$

*) We note that the asymptotic behavior of all the functions of the inner expansion, as prescribed by the boundary condition (10.6) to (10.8), agrees completely with the behavior found from considering the differential equations for these functions.

differential equation for the function $u_2(N)$

$$\frac{\gamma}{\gamma-1} p_0 (u_0 u_2)'' + \frac{1}{4} N u_2' - \left[\frac{\gamma-1}{4\gamma} \frac{1+u_0^2}{u_0^2} + \left(-\frac{2}{3} + \frac{1}{3\gamma}\right)\right] u_2 = 0 \quad (12.6)$$

boundary conditions

$$u_2(0) = 0, \quad u_2(N) \to \left(\frac{2}{3\gamma} \Psi_{10} U_{00} + U_{10}\right) N^{-2/3-2/3\gamma} \quad \text{for } N \to \infty \quad (12.7)$$

equation for the function

$$y_2' + \frac{\gamma-1}{2\gamma p_0} \frac{1+u_0^2}{u_0^2} u_2 = 0 \quad (12.8)$$

The solution of this equation must satisfy the first condition in (10.8). In the end we can find the value of the function $y_2(0)$.

In this manner, the required shape of the wall, at which the pressure distribution attributed to a semi-infinite plate (Section 11) is realized, has the form

$$y \approx y_1(0)\, \xi^{1/4+1/3\gamma} + y_2(0)\, \xi^{1/12+1/3\gamma} \quad (12.9)$$

We note that this result, in accordance with the estimates of the neglected terms made before, has a relative error of the order $\xi^{-1+2/3}\gamma$ or $\xi^{-1/2}$, while the first approximation contains a relative error of order $\xi^{-1/2+1/3\gamma}$.

13. As an example we calculated the viscous flow field for $\gamma = 1.4$ and $\sigma = 1{,}0$. The values of the parameters defining the shape of the shock as well as the pressure distribution, were taken from the solution obtained in [3], $c = 1.4938$ and $p_0' = 0.6268$.

Integration of Equation (11.4) was carried out by the method of iteration. and was presented in the form

$$\frac{\gamma}{\gamma-1} p_0 u_{0(k)}'' + \left[\frac{\gamma}{\gamma-1} p_0 \frac{u_{0(k-1)}'}{u_{0(k-1)}} + \frac{1}{4} N \frac{1}{u_{0(k-1)}}\right] u_{0(k)}' -$$

$$- \frac{1}{4} \frac{\gamma-1}{\gamma} \frac{1+u_{0(k-1)}}{u_{0(k-1)}^2} u_{0(k)} + \frac{1}{4} \frac{\gamma-1}{\gamma} \frac{1+u_{0(k-1)}}{u_{0(k-1)}^2} = 0 \quad (13.1)$$

and as the first approximation for $u_0(k)$ a linear function was applied.

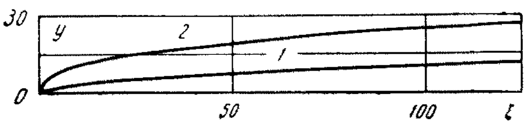

Fig. 2

The convergence of the iterations was defined by the estimate

$$\sum_{i=1}^{n} | u_{0(k-1)}^{i} - u_{0(k)}^{i} | < \varepsilon.$$

$$\varepsilon = 0.0001 \quad (13.2)$$

To integrate Equation (13.1) in each approximation and also to integrate Equations (12.3) and (12.6), the method of iteration was applied.

The calculated results for the inner (viscous) region of the flow are shown in Figs. 1 to 5. The body shape in the first approximation is curve 1, that for the second approximation is curve 2.

As is clear from Fig.1, at sufficiently great distances from the leading edge, the contribution of the term which takes into account the entropy effect in the outer flow core becomes unessential.

The shapes of the front part of the body are shown in Fig.2.

The profiles of velocity, enthalpy, and density are given, respectively, in Figs. 3 to 5 as a function of the veritable $y\xi^{-3/4}$ for values of $\xi = 10$, 100 and 1000. There are also shown the results of the first approximation [3], corresponding to the limiting self-similar solution ($\xi \to \infty$).

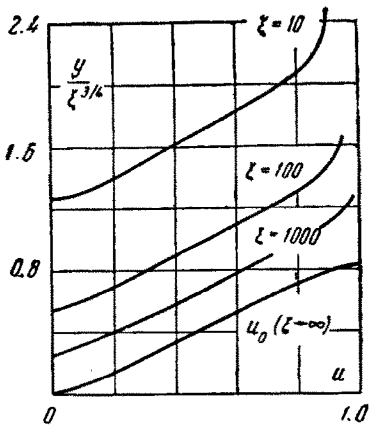

Fig. 3

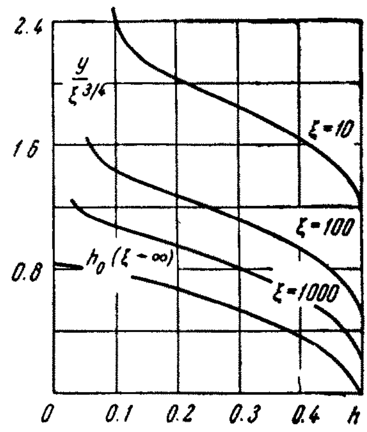

Fig. 4

This study shows that considering the problem of hypersonic viscous gas flow with Mach number $M_\infty \to \infty$ past a slender body as problems in the strong interaction of a boundary layer at the body surface with the inviscid flow field region, permits us to solve this problem to a higher degree of approximation than has been done thus far. Further refinement of the results obtained (determining subsequent terms in the asymptotic expansions) leads to the necessity of considering viscosity in the outer flow field, and considering additional terms in the equations (ordinarily neglected in boundary layer theory) in the inner flow. However, as shown in [1 and 6], such a detailed consideration, strictly speaking, is invalid, since the order of the terms considered in the Navier-Stokes equations will be the same as the order of the Burnett terms, which are not included.

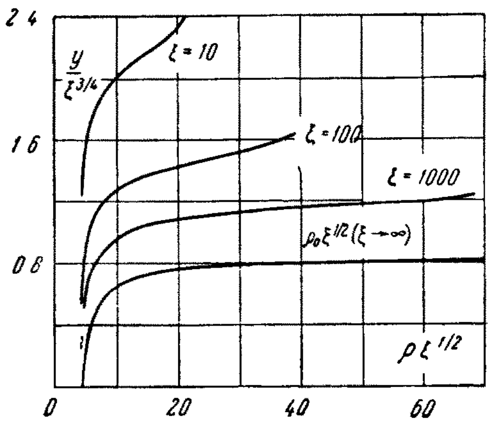

Fig. 5

Using the method of matching inner and outer expansions (as is done in the second part of this paper) together with the method of PLK, apparently, can solve many other problems in which the inner limit of the outer asymptotic solution becomes singular.

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